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THE CONTACT PROBLEM FOR A SHALLOW SHELL WITH A CRACK†

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The contact problem for a shallow shell containing a vertical crack is considered. The solution of the problem satisfies two inequality restrictions describing the mutual non-interpenetration of the shell and a punch, and the condition of non-interpenetration for the crack faces. The purpose of this paper is to investigate a control problem using external loading with an objective functional describing the crack opening. The regularity of the solution is investigated near the tips of the crack. In particular, for a crack with zero opening the solution is shown to below to the class C^∞. The convergence of the solutions of the optimal control problems when the parameters are perturbed is analysed.

1. STATEMENT OF THE PROBLEM

Consider a shallow shell whose median surface occupies the domain $\Omega_\psi = \Omega\backslash\Gamma_\psi$, where $\Omega \subset R^2$ is a bounded domain with smooth boundary Γ and Γ_ψ is the graph of the function $y = \psi(x)$, $x \in [0, 1]$, $(x, y) \in \Omega$. Let $\chi = (W, w)$ be the displacement vector for points of the median surface of the shell, and $W = (w^1, w^2)$. We will introduce the following notation for the components of the strain and stress tensors

$$e_{ij} = \varepsilon_{ij}(W) + k_{ij}w$$
$$\sigma_{11} = e_{11} + ke_{22}, \quad \sigma_{22} = e_{22} + ke_{11}, \quad \sigma_{12} = (1-k)e_{12}$$
$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\right), \quad x_1 \equiv x, \quad x_2 \equiv y$$
$$k = \text{const}, \quad 0 < k < {}^1/_2$$

We assume that the curvature of the shell satisfies $k_{ij} \in C^1(\bar{\Omega}_\psi)$. Here and throughout $i, j = 1, 2$. The energy functional of the shell can be written in the form

$$\Pi_u(\chi) = \tfrac{1}{2}B(w, w) + \tfrac{1}{2}\langle\sigma_{ij}(W),\ e_{ij}(W)\rangle - \langle u, \chi\rangle$$

where $u = (u_1, u_2, u_3)$ is the external load vector, the brackets $\langle\cdot\,,\cdot\rangle$ denote integration over Ω_ψ, and the bilinear form describing the bending properties of the shell has the form

$$B(w, \upsilon) = \int_{\Omega_\psi} (w_{xx}\upsilon_{xx} + w_{yy}\upsilon_{yy} + kw_{xx}\upsilon_{yy} + kw_{yy}\upsilon_{xx} + 2(1-k)w_{xy}\upsilon_{xy})d\Omega_\psi$$

For simplicity we specify the following boundary conditions on the outer boundary

$$w = \partial w/\partial n = W = 0 \quad \text{on} \quad \Gamma$$

The model of the shell under consideration is therefore described by the fact that its median surface is identified with a plane domain, while at the same time the curvature of the shell is not in general zero. Horizontal displacements in this model depend linearly on the distance z from the median surface (see [1])

$$W(z) = W - z\nabla w, \quad |z| \leqslant \delta$$

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where $z = 0$ corresponds to the median surface. Let $\psi \in H_0^3(0, 1)$, and ν be the normal to the curve $y = \psi(x), x \in (0, 1)$. Then the condition of mutual non-interpenetration for the crack faces can be written as follows:

$$[W - z\nabla w]\nu \geq 0 \quad \text{on} \quad \Gamma_\psi, \quad |z| \leq \delta, \quad \nu = (-\psi_x, 1)/\sqrt{1+\psi_x^2}$$

where $[V] = V^+ - V^-$ is the jump in the function V, while $V^\pm$ correspond to the positive and negative directions of ν. We can write this condition in the equivalent form

$$[W]\nu \geq \delta|[\partial w/\partial\nu]| \quad \text{on} \quad \Gamma_\psi \tag{1.1}$$

We assume that the surface $z = \Phi(x,y)$ describes the shape of the punch, $(x,y) \in \Omega$, $\Phi \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)$. In this case the mutual non-interpenetration condition for the shell and the punch, in the linear approximation, has the form [2, 3]

$$w - W\nabla\Phi \geq \Phi \quad \text{in} \quad \Omega_\psi \tag{1.2}$$

Suppose further that the subspace $H^{1,0}(\Omega_\psi)$ of the Sobolev space $H^1(\Omega_\psi)$ consists of elements which vanish on Γ. Elements from $H^{2,0}(\Omega_\psi)$ vanish similarly together with their derivatives on Γ, $H^{2,0}(\Omega_\psi) \subset H^2(\Omega_\psi)$. We denote by $H(\Omega_\psi)$ the space $H^{1,0}(\Omega_\psi) \times H^{1,0}(\Omega_\psi) \times H^{2,0}(\Omega_\psi)$ and introduce the set of admissible displacements of the shell

$$K_\delta = \{(W, w) \in H(\Omega_\psi) | \quad (W, w) \text{ satisfy } (1.1), (1.2)\}$$

Here inequalities (1.1), (1.2) are assumed to be satisfied almost everywhere in the Lebesgue sense on Γ_ψ and in Ω_ψ. We assume that $\Phi < 0$ on Γ, so that the set K_δ is non-empty. The equilibrium problem for a shallow shell with a solution satisfying the non-interpenetration conditions (1.1), (1.2) can be formulated variationally

$$\inf_{\chi \in K_\delta} \Pi_u(\chi) \tag{1.3}$$

Because of the convexity and differentiability of the functional Π_u on $H(\Omega_\psi)$ problem (1.3) is equivalent to the variational inequality

$$\Pi'_u(\chi)(\bar{\chi} - \chi) \geq 0, \quad \chi \in K_\delta, \quad \forall \bar{\chi} \in K_\delta$$

where $\Pi'_u(\chi)$ is the derivative of the functional Π_u at the point χ. This inequality has the form

$$\begin{aligned} &B(w, \bar{w} - w) + \langle k_{ij}\sigma_{ij}, \bar{w} - w\rangle + \langle \sigma_{ij}, \varepsilon_{ij}(\bar{W} - W)\rangle - \langle u, \bar{\chi} - \chi\rangle \geq 0 \\ &\chi \in K_\delta, \quad \forall \bar{\chi} = (\bar{W}, \bar{w}) \in K_\delta \end{aligned} \tag{1.4}$$

It can be proved that the functional Π_u is coercive on $H(\Omega_\psi)$. Using the weak semicontinuity from below of this functional, we verify that a solution of the equilibrium problem (1.4) exists. It will be unique.

We shall investigate the problem of controlling the external loads with an objective functional describing the crack opening [4]

$$J_\delta(u) = \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi$$

where $\chi = \chi(u)$ is the solution of the variational equation (1.4).

Let $U \subset [L^2(\Omega)]^3$ be a convex, closed and bounded set. The problem of finding the crack with the least opening can be formulated as follows:

$$\inf_{u \in U} J_\delta(u) \tag{1.5}$$

Here and below we emphasize the dependence of the objective functional on δ, because later we shall investigate the convergence of the solutions of problem (1.5) when $\delta \to 0$.

Suppose that δ is fixed for the time being. We shall prove that a solution of the optimal control program

(1.5), (1.4) exists. We choose a minimizing sequence $u_m \in U$. It is of course bounded in $L^2(\Omega)$, and so we can assume that

$$u_m \to u \text{ weakly in } L^2(\Omega), \quad u \in U \tag{1.6}$$

For every m one can find a unique solution $\chi_m \in K_\delta$ of the problem

$$\Pi'_{u_m}(\chi_m)(\bar{\chi} - \chi_m) \geqslant 0 \quad \forall \bar{\chi} \in K_\delta \tag{1.7}$$

Fixing the test function $\bar{\chi}$, we derive the estimate

$$\|\chi_m\|_{H(\Omega_\psi)} \leqslant c$$

which is uniform with respect to m.

Having by necessity to choose the sequence, we assume that as $m \to \infty$

$$\chi_m \to \chi \text{ weakly in } H(\Omega_\psi), \quad \text{strongly in } L^2(\Omega_\psi) \tag{1.8}$$

The convergence of (1.6) and (1.8) enables us to take the limit in (1.7) and thus show that $\chi = \chi(u)$. Moreover, additionally assuming that $\chi_m^\pm \to \chi^\pm$ weakly in $L^1(\Gamma_\psi)$, we obtain

$$\inf_{\bar{u} \in U} J_\delta(\bar{u}) = \lim_{m \to \infty} \inf J_\delta(u_m) \geqslant J_\delta(u) \geqslant \inf_{\bar{u} \in U} J_\delta(\bar{u})$$

This also means that u is a solution of problem (1.5), (1.4). The assertion is proved.

2. THE SMOOTHNESS OF THE SOLUTION

We note that if the crack opening is zero on Γ_ψ, i.e. $[\chi] = 0$, the value of the objective functional $J_\delta(u)$ is zero. We also assume that near Γ_ψ the punch does not interact with the shell. It turns out that in this case the solution $\chi = (W, w)$ of problem (1.4) is infinitely differentiable in a neighbourhood of points of the crack. This property is local, so that a zero opening of the crack near the fixed point guarantees infinite differentiability of the solution in some neighbourhood of this point. Here it is undoubtedly necessary to require appropriate regularity of the curvature k_{ij} and the external forces u. The aim of the following discussions is to justify this fact. Here the external load u is taken to be fixed.

Let $O \subset R^2$ be a bounded domain with smooth boundary γ and outward normal $n = (n_1, n_2)$. We introduce the following notation for the bending moment and shear forces on γ

$$m(w) = k\Delta w + (1-k)\frac{\partial^2 w}{\partial n^2}, \quad t(w) = \frac{\partial}{\partial n}\Delta w + (1-k)\frac{\partial^3 w}{\partial n \partial^2 s}, \quad s = (-n_2, n_1)$$

The quantities $m(w)$ and $t(w)$ can be interpreted as elements from the spaces $H^{-1/2}(\gamma)$ and $H^{-3/2}(\gamma)$, respectively if $w \in H^2(O)$, $\Delta^2 w \in L^2(O)$. Moreover, the following generalized Green's formula holds

$$B_o(w, \upsilon) = \left\langle m(w), \frac{\partial \upsilon}{\partial n} \right\rangle_{\frac{1}{2},\gamma} - \langle t(w), \upsilon \rangle_{\frac{3}{2},\gamma} + \langle \Delta^2 w, \upsilon \rangle_o, \quad \forall \upsilon \in H^2(O) \tag{2.1}$$

The symbol O means that the integration is performed over O, while the brackets $\langle \cdot\,, \cdot \rangle_{p,\gamma}$ denote the duality between $H^{-p}(\gamma)$ and $H^p(\gamma)$. All the conditions, imposed on the operators $m(w)$, $t(w)$ and necessary for the validity of formula (2.1), are verified in [6]. Another Green's formula is also needed. Suppose that $\theta \equiv (\upsilon_1, \upsilon_2) \in L^2(O)$, $\operatorname{div} \theta \in L^2(O)$. Then the quantity θn is defined on the boundary as an element of $H^{-1/2}(\gamma)$, and we have the formula [7]

$$\langle \operatorname{div} \theta, w \rangle_o = \langle \theta n, w \rangle_{\frac{1}{2},\gamma} - \langle \theta, \nabla w \rangle_o, \quad \forall w \in H^1(O) \tag{2.2}$$

We shall investigate the regularity of the solution in a neighbourhood of the crack tip $x^0 \equiv (1, 0)$. Suppose first, that (W, w) is a solution of the equilibrium problem (1.4). We assume that a neighbourhood W of the graph Γ_ψ exists such that for any function $\varphi \in C_0^\infty(W)$ there is an $\varepsilon > 0$ for which

$$\varepsilon\varphi + w - W\nabla\Phi \geqslant \Phi \text{ almost everywhere in } W\backslash\Gamma_\psi \tag{2.3}$$

Condition (2.3) can be interpreted as the absence of contact between the shell and the punch in $W\backslash\Gamma_\psi$.

We smoothly continue the function $\psi(x)$ for $x > 1$, keeping the previous notation. We take an arbitrary function $\varphi \in C_0^\infty(R(x^0))$, where $R(x^0)$ is a circle centred at the point x^0 such that $R(x^0) \subset W$. Then

$$[\partial\varphi / \partial\nu] = 0 \quad \text{on} \quad R(x^0) \cap \Gamma_\psi$$

From what has been said, for small $\varepsilon > 0$ the function $(W, \varepsilon\varphi + w)$ belongs to the set K_δ. Outside $R(x^0)$ the function φ can be taken to be zero. We now substitute $(W, \varepsilon\varphi + w)$ into (1.4). We arrive at the inequality

$$B_+(w, \varphi) + B_-(w, \varphi) + \langle k_{ij}\sigma_{ij}, \varphi\rangle \geqslant \langle u_3, \varphi\rangle \tag{2.4}$$

The plus and minus subscripts denote integration over O^+ and O^-, respectively, where $O^+ = R(x^0) \cap \{y > \psi(x)\}$, and similarly for O^-. The boundaries of the domains $O^\pm$ are denoted by $\gamma^\pm$. Note that when (2.3) holds, the equation

$$\Delta^2 w + k_{ij}\sigma_{ij} = u_3 \tag{2.5}$$

is satisfied in $W\backslash\Gamma_\psi$ in the distribution sense.

In order to verify this, it is sufficient to substitute test functions of the form $\chi + \varepsilon\theta$ into (1.4), where θ is an infinitely differentiable function with support $W\backslash\Gamma_\psi$, and ε is a small quantity. Thus, applying Green's formula (2.1) to $B_\pm(w, \varphi)$ in (2.4) and using Eq. (2.5), we obtain

$$\left\langle m(w), \frac{\partial\varphi}{\partial n^-}\right\rangle_{1/2,\gamma^-} - \langle t(w), \varphi\rangle_{3/2,\gamma^-} + \left\langle m(w), \frac{\partial\varphi}{\partial n^+}\right\rangle_{1/2,\gamma^+} - \langle t(w), \varphi\rangle_{3/2,\gamma^+} \geqslant 0 \tag{2.6}$$

Note that in view of the smoothness of the solution the function $\Delta^2 w + k_{ij}\sigma_{ij} - u_3$ is zero almost everywhere in $W\backslash\Gamma_\psi$, and so the integral over the domain vanishes.

Below, ν will also denote the normal to the continued graph $\tilde{\Gamma}_\psi$ of the function $\psi(x)$. Using the arbitrariness and finiteness of φ in $R(x^0)$, from (2.6) we find

$$\langle [m(w)], \partial\varphi / \partial n\rangle_{1/2,\gamma} = 0, \quad \langle [t(w)], \varphi\rangle_{3/2,\gamma} = 0, \quad \forall\varphi \in C_0^\infty(R(x^0)) \tag{2.7}$$

where γ can be taken to be either γ^+ or γ^-. The proven identities (2.7) mean that

$$[m(w)] = 0, \quad [t(w)] = 0 \quad \text{on} \quad \tilde{\Gamma}_\psi \tag{2.8}$$

When conditions (2.3) are satisfied we also have the following distribution equations

$$-\partial\sigma_{ij}/\partial x_j = u_i \quad \text{in} \quad W\backslash\Gamma_\psi \tag{2.9}$$

This is proved simultaneously with (2.5).

Suppose that the function $\theta \equiv (\theta_1, \theta_2)$ belongs to $C_0^\infty(\Omega)$ and has support in $R(x^0)$. Then, as before, for small $\varepsilon > 0$ we have $(W + \varepsilon\theta, w) \in K_\delta$. We substitute $(W + \varepsilon\theta, w)$ into (1.4) as a test function. We obtain

$$\langle\sigma_{ij}, \varepsilon_{ij}(\theta)\rangle_+ + \langle\sigma_{ij}, \varepsilon_{ij}(\theta)\rangle_- \geqslant \langle u_i, \theta_i\rangle$$

Using Green's formula (2.2), it follows from this that

$$-\langle [\sigma_{ij}\nu_j], \theta_i\rangle_{1/2,\gamma} - \langle\partial\sigma_{ij} / \partial x_j, \theta_i\rangle_+ - \langle\partial\sigma_{ij} / \partial x_j, \theta_i\rangle_- \geqslant \langle u_i, \theta_i\rangle$$

where one can take either γ^+ or γ^- to be γ. Bearing in mind Eq. (2.9), the relation obtained gives

$$\langle [\sigma_{ij}\nu_j], \theta_i\rangle_{1/2,\gamma} = 0, \qquad \forall\theta \in C_0^\infty(R(x^0))$$

i.e.

$$[\sigma_{ij}\nu_j]=0 \quad \text{on} \quad \tilde{\Gamma}_\psi \tag{2.10}$$

The established properties (2.8) and (2.10) enable us to investigate the regularity of the solution in a neighbourhood of the crack tip x^0 in the case when there is no contact between the shell and the punch near to x^0, and the crack opening is zero.

Theorem 1. Suppose that $k_{ij}, u \in C^\infty(R(x^0))$, that condition (2.3) is satisfied, and that $[\chi] = 0$ on $R(x^0) \cap \Gamma_\psi$. Then $\chi \in C^\infty(R(x^0))$.

Proof. We shall show that Eq. (2.5) is satisfied in the distribution sense in $R(x^0)$. The condition of the theorem and inequality (1.1) ensure the validity of $[\partial w/\partial\nu] = 0$ on $R(x^0) \cap \Gamma_\psi$. Bearing in mind that $w \in H^2(O^\pm)$ and that $[w] = 0$ on $R(x^0) \cap \Gamma_\psi$, we conclude that $w \in H^2(R(x^0))$.

Note that Eq. (2.5) is satisfied in $O^\pm$, and so $\Delta^2 w \in L^2(O^\pm)$.

Let the brackets $(\cdot, \varphi)$ denote the action of the distribution on the element φ. We choose $\varphi \in C_0^\infty(R(x^0))$. Using formula (2.1) we have

$$(\Delta^2 w, \varphi) = B_+(w,\varphi) + B_-(w,\varphi) = -\langle[m(w)], \partial\varphi/\partial\nu\rangle_{1/2,\gamma} + \langle[t(w)],\varphi\rangle_{3/2,\gamma} +$$
$$+(\Delta^2 w, \varphi)_+ + (\Delta^2 w, \varphi)_-$$

The jumps $[m(w)]$, $[t(w)]$ are zero, from which the necessary equations that prove the assertion follow

$$(\Delta^2 w + k_{ij}\sigma_{ij} - u_3, \varphi) = \langle\Delta^2 w + k_{ij}\sigma_{ij} - u_3, \varphi\rangle_+ + \langle\Delta^2 w + k_{ij}\sigma_{ij} - u_3, \varphi\rangle_- = 0,$$
$$\forall\varphi \in C_0^\infty(R(x^0))$$

We shall now show that Eq. (2.9) is satisfied in $R(x^0)$. Because $[W] = 0$ on $R(x^0) \cap \Gamma_\psi$ and $W \in H^1(O^\pm)$, we have $W \in H^1(R(x^0))$.

Consequently, $\sigma_{ij} \equiv \sigma_{ij}(\chi) \in L^2(R(x^0))$. From the validity of Eqs (2.9) in $O^\pm$, we conclude that $\partial\sigma_{ij}/\partial x_j \in L^2(O^\pm)$. This means that one can apply Green's formula (2.2) to the domains $O^\pm$.

Let $\varphi \in C^\infty{}_0(R(x^0))$. We have

$$-(\partial\sigma_{ij}/\partial x_j + u_i, \varphi) = \langle\sigma_{ij}, \partial\varphi/\partial x_j\rangle_+ + \langle\sigma_{ij}, \partial\varphi/\partial x_j\rangle_- - (u_i, \varphi) =$$

$$= -\langle[\sigma_{ij}\nu_j], \varphi\rangle_{1/2,\gamma} - \langle\partial\sigma_{ij}/\partial x_j + u_i, \varphi\rangle_+ - \langle\partial\sigma_{ij}/\partial x_j + u_i, \varphi\rangle_- = 0 \tag{2.11}$$

However, the jumps $[\sigma_{ij}\nu_j]$ are zero, and Eqs (2.9) are satisfied in $O^\pm$. Hence the right-hand side of (2.11) vanishes, which implies the validity of

$$-\partial\sigma_{ij}/\partial x_j = u_i \quad \text{in} \quad R(x^0) \tag{2.12}$$

in the distribution sense. Equations (2.12) can be written as linear equations in two-dimensional theory of elasticity

$$L(W) = F \quad \text{in} \quad R(x^0) \tag{2.13}$$

with right-hand side $F = (f_1, f_2)$, where $f_1 = u_1 + (k_{11}w + kk_{22}w)_x + (k_{12}w)_y$ and f_2 is defined similarly. Moreover, Eq. (2.5) can be conveniently represented in the form

$$\Delta^2 w = u_3 - k_{ij}\sigma_{ij} \quad \text{in} \quad R(x^0) \tag{2.14}$$

The right-hand side of Eq. (2.13) belongs to $H^1(R(x^0))$ and the right-hand side of (2.14) belongs to $L^2(R(x^0))$. Applying in turn the results on the internal regularity of the solutions of Eqs (2.13) and (2.14) [5, 9], we obtain the necessary inclusion

$$\chi = (W, w) \in C^\infty(R(x^0)) \tag{2.15}$$

The theorem is proved.

We will make a number of remarks. For the inclusion (2.15) to be valid it is sufficient only to require that (2.3) is satisfied in $R(x^0)\backslash\Gamma_\psi$ for $\varphi \in C_0^\infty(R(x^0))$.

According to inclusion theorems the function w is continuous in $\bar{\Omega}_\psi$. Hence if $\nabla\Phi \equiv 0$ in some neighbourhood W of the graph Γ_ψ and $w > \Phi$ in W (and, in particular, $w^\pm > \Phi$ on Γ_ψ), then condition (2.3) is obviously satisfied.

If x^0 is an internal point of the crack, i.e. $x^0 \in \Gamma_\psi\backslash\partial\Gamma_\psi$, if condition (2.3) is satisfied and if $[\chi] = 0$ near x^0, then the corresponding assertion on the infinite differentiability of χ can be proved more simply.

We note that there have been investigations [10–12] of the asymptotic properties of solutions of the equations of the theory of elasticity and the biharmonic equation near the crack tip. Problems of choosing so-called extremal crack shapes have also been investigated [13, 14].

3. THE CONVERGENCE OF THE SOLUTIONS AS $\delta \to 0$

We consider the limiting case corresponding to $\delta = 0$ in (1.1). A restriction obtained in this manner corresponds to the condition of mutual non-interpenetration of the sides of the crack without including the thickness of the shell. We note that in taking full account of the thickness one must bearing mind that the stresses σ_{ij} and the moments $m(w)$ and shear forces $t(w)$ depend on δ. Thus $\delta = 0$ in (1.1) carries the implication that the thickness of the shell is taken to be fixed, and the non-interpenetration conditions on the crack faces are described approximately.

Thus, in the case under consideration the solution satisfies the following restrictions

$$[W]\nu \geqslant 0 \text{ on } \Gamma_\psi; \quad w - W\nabla\Phi \geqslant \Phi \text{ in } \Omega_\psi \tag{3.1}$$

The set of admissible displacements in this case has the form

$$K_0 = \{(W, w) \in H(\Omega_\psi) \,|\, (W, w) \text{ satisfy conditions (3.1)}\}$$

Here the solution of the problem of minimizing the functional Π_u on the set K_0 is equivalent to the solution of the following variational inequality

$$\Pi_u'(\chi)(\bar{\chi} - \chi) \geqslant 0, \quad \chi \in K_0, \quad \forall \bar{\chi} \in K_0 \tag{3.2}$$

Let the set U be chosen as before. We consider the optimal control problem

$$\inf_{u \in U} J_0(u), \quad J_0(u) = \int_{\Gamma_\psi} \|[\chi]\| d\Gamma_\psi \tag{3.3}$$

where χ is defined in (3.2) for given u. A solution of problem (3.3), (3.2) exists (but we will not dwell on the proof).

We introduce the following notation

$$j_\delta = \inf_{u \in U} J_\delta(u), \quad j_0 = \inf_{u \in U} J_0(u) \tag{3.4}$$

The connection between solutions of problems (3.3), (3.2) and (1.5), (1.4) is characterized by the theorem given below. Let u_δ be the solution of problem (1.5), (1.4), while χ_δ corresponds to u_δ and is given by (1.4).

Theorem 2. Let $\nabla\Phi \equiv 0$ in some neighbourhood W of the graph Γ_ψ. Then one can choose from the sequences u_δ, χ_δ subsequences such that as $\delta \to 0$

$$u_\delta \to u_0 \text{ weakly in } L^2(\Omega); \quad \chi_\delta \to \chi_0 \text{ weakly in } H(\Omega_\psi); \quad j_\delta \to j_0$$

where u_0 is a solution of problem (3.2), (3.2), and χ_0 corresponds to u_0 and is given by (3.2).

Proof. Let $\chi_\delta(u)$ be a solution of the variational inequality (1.4) with given fixed $u \in U$. We take an arbitrary element $\bar{\chi} \in K_{\delta_0}$. Then $\bar{\chi} \in K_\delta$ for all $\delta \leqslant \delta_0$. We substitute $\bar{\chi}$ into (1.4) as a test element. We arrive at the estimate

$$\|\chi_\delta(u)\|_{H(\Omega_\psi)} \leqslant c$$

which is uniform with respect to $\delta \leqslant \delta_0$. Consequently, one can assume that when $\delta \to 0$

$$\chi_\delta(u) \to \tilde{\chi} \text{ weakly in } H(\Omega_\psi) \tag{3.5}$$

$$[\chi_\delta(u)] \to [\tilde{\chi}] \text{ weakly in } L^1(\Gamma_\psi) \tag{3.6}$$

$$\delta|[\partial w_\delta(u)/\partial\nu]| \to 0 \text{ strongly in } L^2(\Gamma_\psi) \tag{3.7}$$

We choose an arbitrary element $\bar{\chi} \in K_0$ and construct, in accordance with the lemma (see below), a sequence $\bar{\chi}_\delta \in K_\delta$ which strongly converges to $\bar{\chi}$ in $H(\Omega_\psi)$. Substituting the $\bar{\chi}_\delta$ as test functions into inequality (1.4), using (3.5) we take the limit $\delta \to 0$. Condition (3.7) ensures the inclusion $\tilde{\chi} \in K_0$. The limiting variational inequality has the form

$$\Pi_u'(\tilde{\chi})(\bar{\chi}-\tilde{\chi}) \geqslant 0, \quad \tilde{\chi} \in K_0, \quad \forall \bar{\chi} \in K_0$$

which means $\tilde{\chi} = \chi(u)$. Here, from (3.6), we obtain

$$J_\delta(u) \to J_0(u), \quad \delta \to 0 \tag{3.8}$$

Suppose that u is now a solution of the optimal control problem (3.3), (3.2). From (3.8) we have

$$j_\delta \leqslant J_\delta(u) \to J_0(u) = j_0$$

Hence

$$\limsup j_\delta \leqslant j_0 \tag{3.9}$$

On the other hand, bearing in mind the boundedness of the set U, we can assume

$$\|u_\delta\|_{L^2(\Omega)} \leqslant c \tag{3.10}$$

uniformly with respect to δ. Then from the variational inequalities

$$\Pi_{u_\delta}'(\chi_\delta)(\bar{\chi}-\chi_\delta) \geqslant 0, \quad \chi_\delta \in K_\delta, \quad \forall \bar{\chi} \in K_\delta \tag{3.11}$$

we derive the estimate

$$\|\chi_\delta\|_{H(\Omega_\psi)} \leqslant c \tag{3.12}$$

uniform in δ.

According to (3.10) and (3.12), we can assume without loss of generality that

$$u_\delta \to u_0 \text{ weakly in } L^2(\Omega)$$

$$\chi_\delta \to \chi_0 \text{ weakly in } H(\Omega_\psi), \text{ strongly in } L^2(\Omega_\psi)$$

$$\delta|[\partial w_\delta(u)/\partial\nu]| \to 0 \text{ strongly in } L^2(\Gamma_\psi)$$

This convergence, and the lemma proved below, enable us to take the limit in equality (3.11) and thus obtain

$$\Pi_{u_0}'(\chi_0)(\bar{\chi}-\chi_0) \geqslant 0, \quad \chi_0 \in K_0, \quad \forall \bar{\chi} \in K_0$$

so that $\chi_0 = \chi(u_0)$. As in the proof of relation (3.8), it can be shown that $J_\delta(u_\delta) \to J_0(u_0)$ and therefore

$$\liminf j_\delta \geqslant J_0(u_0) \tag{3.13}$$

Comparing (3.9) and (3.13), we conclude that u_0 is a solution of the optimal control problem (3.3), (3.2) and $j_\delta \to j_0$. The theorem is proved.

It remains to establish the assertion used in the proof of Theorem 2.

Lemma. Let $\nabla\Phi \equiv 0$ in some neighbourhood W of the graph Γ_ψ. Then for any fixed element $\bar{\chi} \equiv (\bar{W}, \bar{w}) \in K_0$ one can construct a sequence $\bar{\chi}_\delta \equiv (\bar{W}_\delta, \bar{w}_\delta) \in K_\delta$ such that

$$(\bar{W}_\delta, \bar{w}_\delta) \to \bar{W}, \bar{w} \text{ strongly in } H(\Omega_\psi) \tag{3.14}$$

Proof. We construct a function $\tilde{W}$ from the space $[H^{1,0}(\Omega_\psi)]^2$ equal to zero outside W and with the property

$$[\tilde{W}]\nu = |[\partial\bar{w}/\partial\nu]| \quad \text{on } \Gamma_\psi$$

If such a function is constructed, the sequence $(\bar{W}_\delta, \bar{w}_\delta) \equiv (\bar{W} + \delta\tilde{W}, \bar{w})$ will be found. Indeed, the convergence of (3.14) is obvious, and moreover

$$\bar{w}_\delta - \bar{W}_\delta\nabla\Phi \geq \Phi \text{ in } \Omega_\psi; \quad [\bar{W}_\delta]\nu \geq \delta|[\partial\bar{w}_\delta/\partial\nu]| \quad \text{on } \Gamma_\psi$$

We therefore choose a simply connected domain O, $\bar{O} \subset \Omega$ with smooth boundary γ such that Γ_ψ is a part of γ, and the outward normal $n = (n_1, n_2)$ to γ coincides with ν on Γ_ψ. We put $g = -|[\partial\bar{w}/\partial n]|$. Then $g \in H^{1/2}(\gamma)$, with $g \equiv 0$ outside Γ_ψ. Since the components of the normal n belong to $C^1(\gamma)$, we have $gn \in [H^{1/2}(\gamma)]^2$. Hence a function $W^0 \in [H^1(O)]^2$ exists such that [5] $W^0 = gn$ on γ.

We put $W^0 \equiv 0$ outside O. Let φ be an infinitely differentiable function on Ω such that $\varphi = 1$ on Γ_ψ and $\varphi \equiv 0$ outside W. The required function $\tilde{W}$ is obtained as follows: $W = \varphi W^0$.

The lemma is proved.

In conclusion we note that the conditions of Theorem 1 do not, in general, ensure the validity of the inclusion (2.15) for the solution $\chi = (W, w)$ of problem (3.2). This is related to the fact that in the case of problem (3.2) the jump $[\partial w/\partial\nu]$ is not, in general, zero on $\Gamma_\psi \cap R(x^0)$, and hence when $[\chi] = 0$ one cannot assert that $w \in H^2(R(x^0))$.

REFERENCES

1. VOL'MIR A. S., *Non-linear Dynamics of Plates and Shells*. Nauka, Moscow, 1972.
2. KHLUDNEV A. M., A variational approach to the problem of the contact of a shallow shell with a rigid body. In *Partial Differential Equations (Proceedings of the S. L. Sobolev Conference)*, No. 2, pp. 109–114, Sib. Otd. Akad. Nauk SSSR, Novosibirsk, 1981.
3. L'VOV G. I., Variational formulation of the contact problem for linearly-elastic and physical non-linear shallow shells. *Prikl. Mat. Mekh.* **46**, 5, 841–846, 1982.
4. GOL'DSHTEIN R. V. and YENTOV V. M., *Qualitative Methods in the Mechanics of Continuous Media*. Nauka, Moscow, 1989.
5. LYONS J.-L. and MAGENES E., *Inhomogeneous Boundary-value Problems and their Applications*. Mir, Moscow, 1971.
6. JOHN O. and NAUMANN J., On regularity of variational solutions of the von Karman equations. *Math. Nachr.* **71,** 23–26, 1976.
7. TEMAM R., *The Navier–Stokes Equations: Theory and Numerical Analysis*. Mir, Moscow, 1981.
8. MIKHAILOV V. P., *Partial Differential Equations*. Nauka, Moscow, 1976.
9. FICHERA G., *Existence Theorems in the Theory of Elasticity*. Mir, Moscow, 1974.
10. KONDRAT'YEV V. A., KOPACHEK I. and OLEINIK O. A., On the behaviour of generalized solutions of second-order elliptic equations and systems of the theory of elasticity in the neighbourhood of a boundary point. In *Proceedings of the I. G. Petrovskii Seminar*, 8, pp. 135–152, Izd. Mosk. Gos. Univ. 1982.
11. OLEINIK O. A., KONDRAT'YEV V. A. and KOPACHEK I., On the asymptotic properties of solutions of the biharmonic equation. *Differents. Uravn.* **17**, 10, 1886–1899, 1981.
12. MOROZOV N. F., *Mathematical Problems in the Theory of Cracks*. Nauka, Moscow, 1984.
13. KHLUDNEV A. M., Extremal shapes of cuts in a plate in contact with a rigid punch. In *Boundary-value Problems for Non-Classical Equations in Mathematical Physics*. Nauka, Novosibirsk, 1989.
14. KHLUDNEV A. M., On extremal shapes of cuts in a plate. *Izv. Ross. Akad. Nauk, MTT* 1, 170–176, 1992.

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